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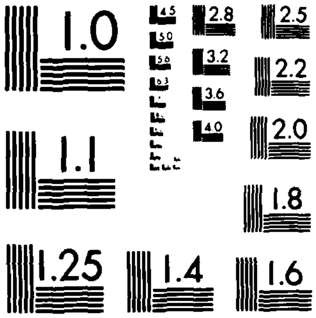
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CENTER FOR STOCHASTIC PROCESSES

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The Asymptotic Behavior of the Likelihood Ratio Statistic for Testing a Shift in Mean in a Sequence of Independent Normal Variates

by

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and

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Let $X_1,\ldots,X_n$ be an independent sequence of random variables such that $X_1,\ldots,X_r \sim$ iid $N(\mu,\sigma^2)$ and $X_{r+1},\ldots,X_n \sim$ iid$(\mu+\theta,\sigma^2)$ where μ, $\theta\neq 0$, σ^2 and r are unknown parameters. The asymptotic properties of the likelihood ratio in testing H_0: $r=n$ (no change point) vs. H_1: $r<n$ are derived. It is shown, using a result of Darling and Erdös, that the likelihood ratio, suitably normalized and under H_0 converges in distribution to the double exponential extreme value distribution. The asymptotic operating characteristics of the likelihood ratio test are studied and comparisons are made between the likelihood ratio test and a Bayesian test.

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The Asymptotic Behavior of the Likelihood Ratio Statistic for Testing a Shift in Mean in a Sequence of Independent Normal Variates

by

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ABSTRACT

Let $X_1, \ldots, X_n$ be an independent sequence of random variables such that $X_1, \ldots, X_r \sim$ iid $N(\mu, \sigma^2)$ and $X_{r+1}, \ldots, X_n \sim$ iid $N(\mu + \theta, \sigma^2)$ where μ, $\theta \neq 0$, σ^2 and r are unknown parameters. The asymptotic properties of the likelihood ratio in testing H_o: $r = n$ (no change point) vs. H_1: $r < n$ are derived. It is shown, using a result of Darling and Erdös, that the likelihood ratio, suitably normalized and under H_o, converges in distribution to the double exponential extreme value distribution. The asymptotic operating characteristics of the likelihood ratio test are studied and comparisons are made between the likelihood ratio test and a Bayesian test.

Running head: Testing a shift in mean.

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1. Introduction

~~We~~ This document considers the problem of testing a sequence of independent and normally distributed random variables with common mean against alternatives involving a shift in the mean at an unknown time point. Specifically, let $X_1, \ldots, X_n$ be an observed independent sequence such that $X_1, \ldots, X_r \sim$ iid $N(\mu, \sigma^2)$ and $X_{r+1}, \ldots, X_n \sim$ iid $N(\mu + \theta, \sigma^2)$ where μ, $\theta \neq 0$ and $r(1 \leq r \leq n)$ are unknown and σ is either known or unknown. We want to test H_o: $r = n$ vs. H_1: $r < n$. Chernoff and Zacks (1964) derived, by assuming $\theta > 0$, a one-sided test statistic through a Bayesian argument. Their results were generalized to the one-parameter exponential family by Kander and Zacks (1966), and to two-sided test statistics by Gardner (1969) and MacNeill (1974). The asymptotic operating characteristics of these Bayesian tests were also studied by Gardner (1969) and MacNeill (1974). Sen and Srivastava (1975) derived various likelihood ratio tests and compared them to Bayesian procedures using computer simulations. Hawkins (1977) discussed heuristically the asymptotic null distribution of the likelihood ratio test. Unfortunately, his reasoning is not entirely correct (though the leading term is of the right order).

The purpose of this paper is to study the asymptotic operating characteristics of the likelihood ratio test. In the next section, ~~we~~ the authors derive, by invoking a result of Darling and Erdös (1956), the asymptotic null distribution of the likelihood ratio test, which is related to the extreme value behavior of the Ornstein-Uhlenbeck process. In Section 3, ~~we study~~ studies the asymptotic operating characteristics of the likelihood ratio test and make comparisons between likelihood ratio and Bayesian tests.

2. Asymptotic null distribution of the likelihood ratio test

Throughout this paper, we assume σ is known since the case of σ unknown is asymptotically equivalent to that of σ known (see Remark 2.4 at the end of this section). So without loss of generality we set $\sigma = 1$.

Given a change in mean occurring between r and r + 1, the maximum likelihood is proportional to $\exp\{-\frac{1}{2}\left[\sum_{i=1}^{r}(X_i - \bar{X}_r)^2 + \sum_{i=r+1}^{n}(X_i - \bar{X}_r')^2\right]\}$ where $\bar{X}_r$ denotes the sample mean of the first r observations and $\bar{X}_r'$ the sample mean of the last n - r observations. So the (generalized) minus log likelihood ratio of H_o vs. H_1 is proportional to (see for example Sen and Srivastava, 1975)

$$T_n^2 \equiv \max_{1\le r\le n-1}\left[\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 - \sum_{i=1}^{r}(X_i - \bar{X}_r)^2 - \sum_{i=r+1}^{n}(X_i - \bar{X}_r')^2\right] \tag{2.1}$$

$$= \max_{1\le r\le n-1} \frac{\left(\frac{S_r}{\sqrt{n}} - \frac{r}{n}\frac{S_n}{\sqrt{n}}\right)^2}{(\frac{r}{n})(1 - \frac{r}{n})}$$

$$= \left[\max_{1\le r\le n-1} \frac{\left|\frac{S_r}{\sqrt{n}} - \frac{r}{n}\frac{S_n}{\sqrt{n}}\right|}{((\frac{r}{n})(1 - \frac{r}{n}))^{\frac{1}{2}}}\right]^2$$

where $S_r = X_1 + \ldots + X_r$. Suppose $\{W(t);\ 0 \le t < \infty\}$ is a standard Brownian motion. Since under H_o, $\{(S_r - r\mu)/\sqrt{n};\ 1 \le r \le n\} \overset{\mathcal{D}}{=} \{W(\frac{r}{n});\ 1 \le r \le n\}$, we have

$$T_n \equiv \max_{1\le r\le n-1} \left|\frac{S_r}{\sqrt{n}} - \frac{r}{n}\frac{S_n}{\sqrt{n}}\right| \Big/ ((\frac{r}{n})(1 - \frac{r}{n}))^{\frac{1}{2}} \tag{2.2}$$

$$\overset{\mathcal{D}}{=} \max_{nt=1,\ldots,n-1} |W(t) - tW(1)|/(t(1-t))^{\frac{1}{2}}$$

$$= \max_{nt=1,\ldots,n-1} |W_o(t)|/(t(1-t))^{\frac{1}{2}}$$

where "$\overset{\mathcal{D}}{=}$" means "equal in distribution" and $W_o(t) = W(t) - tW(1)$, (i.e. Brownian bridge). Throughout, we shall use the convention that $\max\limits_{nt=1,\ldots,n-1} = \max\limits_{1\le nt\le n-1}$.

<u>Theorem 2.1</u> Under H_o,

$$\lim_{n\to\infty} P(a_n^{-1}(T_n - b_n) \leq x) = \exp(-2\pi^{-\frac{1}{2}} e^{-x}), \quad -\infty < x < \infty$$

where $a_n = (2\,\ell n_2\, n)^{-\frac{1}{2}}$, $b_n = a_n^{-1} + 2^{-1} a_n\, \ell n_3\, n$ and ℓn_k is the k-th iterated logarithm.

We need the following lemmas to prove the theorem.

<u>Lemma 2.2</u> (Theorem 2 of Darling and Erdös, 1956)

$$\lim_{n\to\infty} P\left(a_n^{-1}\left(\max_{k=1,\ldots,n}\left|\frac{W(k)}{\sqrt{k}}\right| - b_n\right) \leq x\right) = \exp(-\pi^{-\frac{1}{2}} e^{-x}), -\infty < x < \infty.$$

<u>Lemma 2.3</u>

$$\max_{1\leq nt\leq [n/\ell n\, n]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} - \max_{1\leq nt\leq [n/\ell n\, n]} \frac{|W(t)|}{\sqrt{t}} = o_p(a_n)$$

<u>Proof</u>. For n large and $t \leq 1/\ell n\, n$, we have

$$\left|\frac{|W(t) - tW(1)|}{\sqrt{t(1-t)}} - \frac{|W(t)|}{\sqrt{t}}\right| \leq \left|\frac{W(t) - tW(1)}{\sqrt{t(1-t)}} - \frac{W(t)}{\sqrt{t}}\right|$$

$$\leq \left|\frac{W(t)}{\sqrt{t}}\left(\frac{1}{\sqrt{1-t}} - 1\right)\right| + \left|\sqrt{\frac{t}{1-t}}\, W(1)\right|$$

$$\leq \sqrt{t}|W(t)| + 2\sqrt{t}|W(1)|$$

$$\leq (\ell n\, n)^{-\frac{1}{2}}(|W(t)| + 2|W(1)|).$$

Therefore,

$$\left|\max_{1\leq nt\leq [n/\ell n\, n]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} - \max_{1\leq nt\leq [n/\ell n\, n]} \frac{|W(t)|}{\sqrt{t}}\right|$$

$$\leq \max_{1\leq nt\leq [n/\ell n\, n]} \left|\frac{|W_o(t)|}{\sqrt{t(1-t)}} - \frac{|W(t)|}{\sqrt{t}}\right|$$

$$\leq (\ell n\, n)^{-\frac{1}{2}} \max_{1\leq nt\leq [n/\ell n\, n]} (|W(t)| + 2|W(1)|)$$

$$= 0_p((\ell n\, n)^{-\frac{1}{2}}).$$

□

Lemma 2.4

$$\max_{[n/\ln n]\le nt\le[n/2]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} = O_p((\ln_3 n)^{1/2})$$

Proof. By the law of the iterated logarithm, for t small enough,

$$|W(t)| < 2\sqrt{2t \ln_2 t^{-1}}, \text{ W.P.1.}$$

It follows that as $s \to 0+$,

$$\max_{t\in[s,1/2]} \frac{|W(t)|}{\sqrt{t}} = O_p((\ln_2 s^{-1})^{1/2})$$

so that

$$\max_{[n/\ln n]\le nt\le[n/2]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} \le \max_{[n/\ln n]\le nt\le[n/2]} \frac{|W(t)|}{\sqrt{t}} + \max_{[n/\ln n]\le nt\le[n/2]} \sqrt{\frac{t}{1-t}}\,|W(1)|$$

$$= O_p((\ln_3 n)^{1/2}) + O_p(1). \qquad \square$$

Lemma 2.5

$$\lim_{n\to\infty} P(a_n^{-1}(\max_{1\le nt\le[n/\ln n]} \frac{|W(t)|}{\sqrt{t}} - b_n) \le x) = \exp(-\pi^{-1/2} e^{-x}).$$

Proof. Since $\{\frac{W(t)}{\sqrt{t}};\ t = \frac{1}{n}, \ldots, \frac{[n/\ln n]}{n}\} \overset{\mathcal{D}}{=} \{\frac{W(t)}{\sqrt{t}};\ t = 1, \ldots, [n/\ln n]\}$,

it follows from Lemma 2.2 that

$$P(\max_{1\le t\le[n/\ln n]} \frac{|W(t)|}{\sqrt{t}} \le a_{[n/\ln n]}x + b_{[n/\ln n]}) \to \exp(-\pi^{-1/2} e^{-x}).$$

But,

$$a_{[n/\ln n]} = a_n + o(a_n) \text{ and } b_{[n/\ln n]} = b_n + o(a_n)$$

which establishes the lemma by the convergence to types result. $\square$

Lemma 2.6

$$\max_{1\le nt\le[n/2]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} - \max_{1\le nt\le[n/\ln n]} \frac{|W(t)|}{\sqrt{t}} = o_p(a_n).$$

Proof. From Lemmas 2.3 and 2.5, we have

$$(2\,\ell n_2\, n)^{-\frac{1}{2}} \max_{1 \le nt \le [n/\ell n\, n]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} \xrightarrow{n\to\infty} 1 \text{ in probability.}$$

This property and Lemma 2.4 together imply

$$P\left(\max_{[n/\ell n\, n] \le nt \le [n/2]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} \ge \max_{1 \le nt \le [n/\ell n\, n]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} \right) \xrightarrow{n\to\infty} 0.$$

Now, applying Lemma 2.3 completes the proof. □

Lemma 2.7

$$\max_{1 \le n(1-t) \le [n/2]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} - \max_{1 \le n(1-t) \le [n/\ell n\, n]} \frac{|W(t) - W(1)|}{\sqrt{1-t}} = o_p(a_n).$$

Proof. This follows from Lemma 2.6 and the symmetry of the Brownian bridge $W_o(t)$ with respect to $t = \frac{1}{2}$; i.e. $\{W_o(t);\ 0 \le t \le 1\} \overset{\mathcal{D}}{=} \{W_o(1-t);\ 0 \le t \le 1\}$. □

Proof of Theorem 2.1. We have

$$P(T_n \le a_n x + b_n | H_o) = P\left(\max_{1 \le nt \le n-1} \frac{|W_o(t)|}{\sqrt{t(1-t)}} \le a_n x + b_n \right)$$

$$= P\left(\max_{1 \le nt \le [n/2]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} \le a_n x + b_n,\ \max_{1 \le n(1-t) \le [n/2]} \frac{|W_o(t)|}{\sqrt{t(1-t)}} \le a_n x + b_n \right)$$

which by Lemmas 2.6 and 2.7 is equal to

$$P\left(\max_{1 \le nt \le [n/\ell n\, n]} \frac{|W(t)|}{\sqrt{t}} \le a_n x + b_n,\ \max_{1 \le n(1-t) \le [n/\ell n\, n]} \frac{|W(t)-W(1)|}{\sqrt{1-t}} \le a_n x + b_n \right) + o(1)$$

$$\to \exp(-2\pi^{-\frac{1}{2}} e^{-x}).$$

by Lemma 2.5 and the independence of the increments of Brownian motion. □

Remark 2.1. From extreme value theory, it is known that the extreme value of a Gaussian sequence converges to its limit distribution very slowly. Therefore, one must be careful in using the limit distribution as an approximation for finite n.

Remark 2.2. Hawkins (1977) approximated the asymptotic null distribution by that of the maximum of $\ell n\, n$ consecutive observations from a weakly dependent stationary Gaussian sequence. Appealing to a result of Berman (1964), this statistic has the same limit distribution as the maximum of $\ln n$ observations from an iid $N(0, 1)$ sequence. However, since $W(t)/\sqrt{t} = X(\frac{1}{2}\ln t)$ where $X(t)$ is the Ornstein-Uhlenbeck process, the above argument shows that the null distribution is approximated by the extreme value (in continuous time) of the Ornstein-Uhlenbeck process. The normalizing constants for the maximum of this process differ from those of the maximum from an iid $N(0, 1)$ sequence (cf. Leadbetter, Lindgren and Rootzen, 1983) which accounts for the discrepancy between the normalizing constants in Theorem 2.1 and those suggested by Hawkins.

Remark 2.3. The statistic T_n was derived from the two-sided likelihood ratio (i.e. θ could be either positive or negative). If it is given that $\theta \leq 0$ (i.e. downward shift in mean), a statistic derived from the one-sided likelihood ratio is (cf. (2.5) of Sen and Srivastava, 1975)

$$(2.3) \qquad T_n' = \max\left(0, \max_{1\leq r\leq n-1} \frac{\frac{S_r}{\sqrt{n}} - \frac{r}{n}\frac{S_n}{\sqrt{n}}}{\left(\frac{r}{n}\left(1 - \frac{r}{n}\right)\right)^{\frac{1}{2}}}\right).$$

Applying Theorem 1 of Darling and Erdös (1956) and following the pattern of the proof of Theorem 2.1, it can be shown that under H_o,

$$(2.4) \qquad \lim_{n\to\infty} P(a_n^{-1}(T_n' - b_n) \leq x) = \exp(-\pi^{-\frac{1}{2}} e^{-x}), \ -\infty < x < \infty.$$

Remark 2.4. When the common variance σ^2 is unknown, the likelihood ratio statistic of H_o vs. H_1 becomes

$$\min_{1\leq r\leq n-1} \left(\frac{\hat{\sigma}_r^2}{\hat{\sigma}_n^2}\right)^{n/2} = \left(\min_{1\leq r\leq n-1} \frac{n\,\hat{\sigma}_r^2}{n\,\hat{\sigma}_n^2}\right)^{n/2}$$

$$= \left(1 - \frac{1}{n}\left[\max_{1\leq r\leq n-1} \hat{\sigma}_n^{-2}\left\{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 - \sum_{i=1}^{r}(X_i - \bar{X}_r)^2 - \sum_{i=r+1}^{n}(X_i - \bar{X}_r')^2\right\}\right]\right)^{n/2}$$

where $\hat{\sigma}_r^2 = n^{-1}[\sum_{i=1}^{r} (X_i - \bar{X}_r)^2 + \sum_{i=r+1}^{n} (X_i - \bar{X}_r')^2]$. So a test statistic is

$$T_n'' = \frac{\sigma^2}{\hat{\sigma}_n^2} \max_{1 \le r \le n-1} \sigma^{-2}[\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 - \sum_{i=1}^{r} (X_i - \bar{X}_r)^2 - \sum_{i=r+1}^{n} (X_i - \bar{X}_r')^2]$$

where a large value of T_n'' indicates significance. The random variable $\max_{1 \le r \le n-1} \sigma^{-2}[\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 - \sum_{i=1}^{r} (X_i - \bar{X}_r)^2 - \sum_{i=r+1}^{n} (X_i - \bar{X}_r')^2]$ has the same null distribution as the likelihood ratio statistic under the condition of σ being known equal to 1. We also have, under H_o,

$$\frac{\sigma^2}{\hat{\sigma}_n^2} = 1 + 0_p(n^{-\frac{1}{2}})$$

Therefore, the likelihood ratio statistic for both cases of σ known and unknown has the same asymptotic null distribution.

3. Operating characteristics of the likelihood ratio test $\{T_n\}$

We shall compute the local powers of T_n and compare the performance of T_n and the Bayesian statistic $U_n \equiv \frac{1}{n^2} \sum_{r=1}^{n-1} \left[\sum_{i=r}^{n-1} (X_{i+1} - \bar{X}_n)\right]^2$, (see (1.3) of Gardner (1969), assuming the uniform prior distribution on the location of the change point).

Let α be a given significance level. From Theorem 2.1, $\lim_{n\to\infty} P(T_n > c_n) = \alpha$ where

(3.1) $$c_n = b_n - a_n \ \ell n[2^{-1} \pi^{\frac{1}{2}} \ \ell n(1 - \alpha)^{-1}].$$

We consider the alternative $H(t_o, \eta, n)$ that there is a change between $[n\, t_o]$ and $[n\, t_o] + 1$ and the amount of (positive) shift in mean is

(3.2) $$\theta = (n\, t_o\, (1 - t_o))^{-\frac{1}{2}} ((2\ \ell n_2\ n)^{\frac{1}{2}} + \eta + o(1))$$

where t_o and η are two constants with $0 < t_o < 1$. (The results below are exactly the same for negative shifts.)

Theorem 3.1.

$$\lim_{n\to\infty} P(T_n > c_n | H(t_o, \eta, n)) = \alpha + \Phi(\eta)(1 - \alpha)$$

where $\Phi(\cdot)$ is the standard normal cdf.

Proof of Theorem 3.1.

Again, let $W_o(t) = W(t) - tW(1)$ be a Brownian bridge. Under $H(t_o, \eta, n)$,

$$(3.3) \qquad T_n \equiv \max_{1\le r\le n-1} \left|\frac{S_r}{\sqrt{n}} - \frac{r}{n}\frac{S_n}{\sqrt{n}}\right| \Big/ \left(\frac{r}{n}\left(1 - \frac{r}{n}\right)\right)^{\frac{1}{2}}$$

$$\stackrel{\mathcal{D}}{=} \max_{1\le nt\le n-1} |W_o(t) - f_n(t)|/(t(1-t))^{\frac{1}{2}}$$

where

$$(3.4) \qquad \begin{aligned} f_n(t) &= n^{\frac{1}{2}}\,\theta\,(1 - [n\,t_o]/n)t && \text{if } nt \le [n\,t_o], \\ &= n^{\frac{1}{2}}\,\theta\,\frac{[n\,t_o]}{n}(1-t) && \text{if } nt > [n\,t_o]. \end{aligned}$$

Denote by $E_{n,\ell}$ $(1 \le \ell \le 6)$ the event that

$$\left\{\max_{t\in D_{n,\ell}} |W_o(t) - f_n(t)|/(t(1-t))^{\frac{1}{2}} > c_n\right\}$$

where

$$D_{n,1} = \{k/n: 1 \le k \le [n/\ell n\, n]\}$$

$$D_{n,2} = \{k/n: n - [n/\ell n\, n] \le k \le n - 1\}$$

$$D_{n,3} = \{k/n: [n/\ell n\, n] < k < [n\,t_o] - [n(\ell n_3 n)/(\ell n_2\, n)^{\frac{1}{2}}]\}$$

$$D_{n,4} = \{k/n: [n\,t_o] + [n(\ell n_3 n)/(\ell n_2\, n)^{\frac{1}{2}}] < k < n - [n/\ell n\, n]\}$$

$$D_{n,5} = \{k/n: [n\,t_o] - [n(\ell n_3 n)/(\ell n_2\, n)^{\frac{1}{2}}] \le k \le [n\,t_o]\}$$

$$D_{n,6} = \{k/n: [n\,t_o] < k \le [n\,t_o] + [n(\ell n_3 n)/(\ell n_2\, n)^{\frac{1}{2}}]\}.$$

Obviously,

$$P(T_n > c_n | H(t_o, \eta, n)) = P\left(\max_{1\le nt\le n-1} |W_o(t) - f_n(t)|/(t(1-t))^{\frac{1}{2}} > c_n\right)$$

$$= P\left(\bigcup_{\ell=1}^{6} E_{n,\ell}\right).$$

We now break up the proof into a few lemmas.

Lemma 3.2

$$P\left(E_{n,1} \,\Delta \left\{\max_{t\in D_{n,1}} \frac{|W(t)|}{\sqrt{t}} > c_n\right\}\right) = o(1)$$

$$P\left(E_{n,2} \,\Delta \left\{\max_{t\in D_{n,2}} \frac{|W(t) - W(1)|}{\sqrt{1-t}} > c_n\right\}\right) = o(1).$$

(Δ denotes symmetric difference.)

Proof. We only prove the first equation. The second one can be done similarly.

For $1 \le nt \le [n/\ell n\ n]$ and n large, we have

$$\frac{|f_n(t)|}{\sqrt{t(1-t)}} \le 2\,\frac{|f_n(t)|}{\sqrt{t}} = 2(n^{\frac{1}{2}}\,\theta)\left(1 - \frac{[n\ t_o]}{n}\right)\sqrt{t}$$

and using (3.2), this bound is $O((\ell n_2 n\ /\ell n\ n)^{\frac{1}{2}}) = o(a_n)$ uniformly in $t\in D_{n,1}$.

So,

$$\max_{t\in D_{n,1}} \frac{|W_o(t)|}{\sqrt{t(1-t)}} - \max_{t\in D_{n,1}} \frac{|W_o(t) - f_n(t)|}{\sqrt{t(1-t)}} = o_p(a_n)$$

which together with Lemmas 2.3 and 2.5 completes the proof. $\square$

Lemma 3.3

$$P(E_{n,\ell}) = o(1), \quad \ell = 3,4.$$

Proof. We only consider the case $\ell = 3$. From Lemma 2.4,

$$\max_{t\in D_{n,3}} \frac{|W_o(t)|}{\sqrt{t(1-t)}} = 0_p((\ell n_3 n)^{\frac{1}{2}}).$$

Also for n large,

$$\max_{t\in D_{n,3}} \frac{|f_n(t)|}{\sqrt{t(1-t)}} = \max_{t\in D_{n,3}} n^{\frac{1}{2}}\,\theta(1 - [n\ t_o]/n)(t/(1-t))^{\frac{1}{2}}$$

$$= (n^{\frac{1}{2}}\,\theta)(1 - [n\ t_o]/n)(t/(1-t))^{\frac{1}{2}} \text{ at } t = t^* = n^{-1}\left([n\ t_o] - \left[\frac{n\ \ell n_3\ n}{(\ell n_2\ n)^{\frac{1}{2}}}\right]\right)$$

$$= t_o - (\ell n_3 n)/(\ell n_2 n)^{\frac{1}{2}} + 0(n^{-1})$$

$$= [t_o(1-t_o)]^{-\frac{1}{2}}((2\ \ell n_2\ n)^{\frac{1}{2}} + \eta + o(1))(1 - t_o + O(n^{-1}))(t^*/(1-t^*))^{\frac{1}{2}}$$

$$= (2\ \ell n_2\ n)^{\frac{1}{2}} - 2^{-\frac{1}{2}}(1 + o(1))(\ell n_3 n)/(t_o(1-t_o)).$$

Therefore, the lemma follows from the inequality

$$\max_{t \in D_{n,3}} \frac{|W_o(t) - f_n(t)|}{\sqrt{t(1-t)}} \le O_p((\ell n_3\ n)^{\frac{1}{2}}) + (2\ \ell n_2\ n)^{\frac{1}{2}}$$

$$- 2^{-\frac{1}{2}}(1 + o(1))(\ell n_3 n)/(t_o(1-t_o)). \quad \square$$

Lemma 3.4

$$P(E_{n,\ell}\ \Delta\{(t_o(1-t_o))^{\frac{1}{2}}\, n^{\frac{1}{2}}\, \theta - \frac{W_o(t_o)}{\sqrt{t_o(1-t_o)}} > c_n\}) = o(1),\ \ell = 5,\ 6.$$

Proof. Once again we only supply the proof for one case $\ell = 5$. Observe that

$$(3.5) \qquad \max_{t \in D_{n,5}} \frac{f_n(t)}{\sqrt{t(1-t)}} = n^{\frac{1}{2}}\, \theta((1 - t_o)t_o)^{\frac{1}{2}} + O((\ell n_2\ n)^{\frac{1}{2}}/n) \sim (2\ \ell n_2\ n)^{\frac{1}{2}}$$

and since $W_o(t)$ is W.P.1 continuous

$$(3.6) \qquad \max_{t \in D_{n,5}} \left| \frac{W_o(t)}{\sqrt{t(1-t)}} - \frac{W_o(t_o)}{\sqrt{t_o(1-t_o)}} \right| = o_p(1).$$

It follows that $R_n \equiv \max_{t \in D_{n,5}} \frac{f_n(t) - W_o(t)}{\sqrt{t(1-t)}}$ is eventually equal to $\max_{t \in D_{n,5}} \frac{|W_o(t) - f_n(t)|}{\sqrt{t(1-t)}}$. Set $Z_n = (t_o(1-t_o))^{\frac{1}{2}} n^{\frac{1}{2}}\, \theta - W_o(t_o)(t_o(1-t_o))^{-\frac{1}{2}}$. It thus suffices to show that for every $\varepsilon > 0$,

$$(3.7) \qquad P(|Z_n - R_n| > \epsilon) \to 0$$

and

$$(3.8) \qquad \lim_{\epsilon \to 0} \limsup_{n \to \infty} P(c_n - \epsilon \le Z_n \le c_n + \epsilon) = 0.$$

We have

$$\max_{t\in D_{n,5}} \frac{f_n(t)}{\sqrt{t(1-t)}} - \max_{t\in D_{n,5}} \frac{W_o(t)}{\sqrt{t(1-t)}} - Z_n \le R_n - Z_n$$

$$\le \max_{t\in D_{n,5}} \frac{f_n(t)}{\sqrt{t(1-t)}} - \min_{t\in D_{n,5}} \frac{W_o(t)}{\sqrt{t(1-t)}} - Z_n .$$

By (3.5) and (3.6) the two outside terms of this inequality are both $o_p(1) + O((\ell n_2 n)^{\frac{1}{2}}/n) = o_p(1)$ which proves (3.7). As for (3.8), we have $c_n - (t_o(1-t_o))^{\frac{1}{2}}n^{\frac{1}{2}}\theta = -\eta + o(1)$ so that

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} P(c_n - \epsilon \le Z_n \le c_n + \epsilon) = \lim_{\epsilon\to 0} P\left(\eta - \epsilon \le \frac{W_o(t_o)}{\sqrt{t_o(1-t_o)}} \le \eta + \epsilon\right)$$

$$= 0. \qquad \square$$

Continuation of the proof of Theorem 3.1: First it is a straightforward calculation to show

$$P(\{(t_o\,(1-t_o))^{\frac{1}{2}}n^{\frac{1}{2}}\,\theta - W_o(t_o)/(t_o(1-t_o))^{\frac{1}{2}} > c_n\} \,\Delta\, A_n) = o(1) \tag{3.9}$$

where

$$A_n = \{(t_o(1-t_o))^{\frac{1}{2}}\, n^{\frac{1}{2}}\,\theta - \frac{[W(t_o) - W((\ell n\ n)^{-\frac{1}{2}}) - t_o(W(1-(\ell n\ n)^{-\frac{1}{2}}) - W((\ell n\ n)^{-\frac{1}{2}}))]}{(t_o(1-t_o))^{\frac{1}{2}}} > c_n\}.$$

By (3.9), Lemmas 2.5, 3.2-3.4, and the independence of the Brownian increments, we have

$$P(T_n > c_n | H(t_o, \eta, n)) = P(\bigcup_{\ell=1}^{6} E_{n,\ell})$$

$$= P(\{\max_{t\in D_{n,1}} \frac{W(t)}{\sqrt{t}} > c_n\} \cup \{\max_{t\in D_{n,2}} \frac{|W(t) - W(1)|}{\sqrt{1-t}} > c_n\} \cup A_n) + o(1)$$

$$\to \alpha + \Phi(\eta)(1-\alpha). \qquad \square$$

MacNeill (1971) studied the asymptotic behavior of $U_n \equiv n^{-2} \sum_{r=1}^{n-1} (\sum_{i=r}^{n-1} (X_{i+1} - \bar{X}_n))^2$.

We state his results under H_o and under the alternative $H'(t_o, \psi, n)$ that there is a change between $[n\ t_o]$ and $[n\ t_o] + 1$ and the amount of shift in mean is $\psi\ n^{-\frac{1}{2}}$.

Lemma 3.5

(i) Under H_o, $U_n \xrightarrow{\mathcal{D}} \int_o^1 W_o^2(t)dt$

(ii) Under $H'(t_o, \psi, n)$,

$$U_n \xrightarrow{\mathcal{D}} \int_o^1 (W_o(t) + h(t))^2 dt$$

where
$$h(t) = \psi\, t_o\, t \qquad 0 \le t \le 1 - t_o$$
$$= \psi\, (1-t_o)(1-t), \qquad 1 - t_o \le t \le 1$$

We now compare the two tests relative to two sequences of alternatives both of which are approaching H_o (in some sense). For the first comparison, t_o is fixed and a sequence of shifts $\psi(t_o, \alpha, \beta)n^{-1/2}$ converging to 0 is specified so that the sequence of tests $\{U_n\}$ has asymptotic significance α and power $\beta > \alpha$. In other words, if $c(\alpha)$ is such that $P(\int_o^1 W_o^2(t)dt > c(\alpha)) = \alpha$, then $\psi = \psi(t_o, \alpha, \beta)$ is chosen so that $P(\int_o^1 (W_o(t) + h(t))^2 dt > c(\alpha)) = \beta$. Then the sample size $n' = n'(n)$ required in order for the likelihood ratio test to have asymptotic significance α and power β w.r.t. the sequence of shifts $\psi(t_o, \alpha, \beta)n^{-1/2}$ is found by solving $(n'\, t_o(1-t_o))^{-1/2}((2\, \ell n_2\, n')^{1/2} + \eta + o(1)) = \psi(t_o, \alpha, \beta)n^{-1/2}$ (cf. (3.2)) where $\eta = \eta(\alpha, \beta)$ satisfies $\alpha + \Phi(\eta)(1-\alpha) = \beta$. It is easily seen that n' should grow like $(2n\, \ell n_2\, n)/(\psi^2(t_o, \alpha, \beta)t_o(1-t_o))$. This suggests defining the analogue of Pitman efficiency (depending on t_o, α, β) of the sequence of tests $\{T_n\}$ w.r.t. $\{U_n\}$ by $n/n' \sim \psi^2(t_o, \alpha, \beta)t_o(1-t_o)/(2\, \ell n_2\, n)$. Thus for the sequence of alternatives specified above, $\{U_n\}$ is more efficient than $\{T_n\}$ when n is very large (although $\ell n\, \ell n\, n$ grows rather slowly).

For the second comparison, we consider the sequence of alternatives $\{H_{(n)}\}$ where the amount of shift θ is now held constant but $t_o = n^{-1}[n - n^{1-\gamma}] \approx 1 - n^{-\gamma}$, $\frac{1}{2} < \gamma < 1$, is approaching 1. Then for the alternatives $\{H_{(n)}\}$ the sequence of tests $\{T_n\}$ is more powerful than $\{U_n\}$ since

(3.10) $\quad P(U_n > c(\alpha)|H_{(n)}) \to \alpha$

while

(3.11) $\quad P(T_n > c_n|H_{(n)}) \to 1,$

where c_n is given by (3.1).

The proof of these two results is as follows: Let Y_i be iid $N(0, 1)$. Then

$$U_n = n^{-2} \sum_{r=1}^{n-1}\left(\sum_{i=r}^{n-1}(X_{i+1} - \overline{X})\right)^2 \overset{\mathcal{D}}{=} n^{-2} \sum_{r=1}^{n-1}\left(\sum_{i=r}^{n-1}[(Y_{i+1} - \overline{Y}) - g_n(i+1)]\right)^2$$

$$= n^{-2} \sum_{r=1}^{n-1}\left(\sum_{i=r}^{n-1}(Y_{i+1} - \overline{Y})\right)^2 - 2n^{-2} \sum_{r=1}^{n-1}\left(\sum_{i=r}^{n-1}(Y_{i+1} - \overline{Y})\right)\left(\sum_{i=r}^{n-1} g_n(i+1)\right)$$

$$+ n^{-2} \sum_{r=1}^{n-1}\left(\sum_{i=r}^{n-1} g_n(i+1)\right)^2$$

where

$$g_n(i) = \begin{cases} -(1-[n - n^{1-\gamma}]/n)\theta & \text{if } i \leq [n - n^{1-\gamma}]. \\ ([n - n^{1-\gamma}]/n)\theta & \text{if } i > [n - n^{1-\gamma}]. \end{cases}$$

Since $n^{-2} \sum_{r=1}^{n-1}\left(\sum_{i=r}^{n-1}(Y_{i+1} - \overline{Y})\right)^2 \xrightarrow{\mathcal{D}} \int_0^1 W_0^2(t)dt$, it suffices to show

(3.12) $$n^{-2} \sum_{r=1}^{n-1}\left(\sum_{i=r}^{n-1} g_n(i+1)\right)^2 = o(1).$$

We have for $\frac{1}{2} < \gamma < 1$,

$$n^{-2} \sum_{r=1}^{n-1}\left(\sum_{i=r}^{n-1} g_n(i+1)\right)^2 \leq n^{-2}\, n\{n^{-1}(n - [n - n^{1-\gamma}])n|\theta| + (n - [n - n^{1-\gamma}])|\theta|\}^2$$

$$\leq n^{-1}(2n^{1-\gamma}|\theta| + O(1))^2$$

$$\to 0 \text{ as } n \to \infty$$

which establishes (3.12) and thus proves (3.10).

As for (3.11), we have

$$T_n \geq \frac{|W_o(t_o) - f_n(t_o)|}{(t_o(1-t_o))^{\frac{1}{2}}} = \left|\frac{W_o(t_o)}{(t_o(1-t_o))^{\frac{1}{2}}} - \theta\, n^{\frac{1}{2}(1-\gamma)} + o(n^{\frac{1}{2}(1-\gamma)})\right|$$

$$= |\theta| n^{\frac{1}{2}(1-\gamma)} + o_p(n^{\frac{1}{2}(1-\gamma)}),$$

whence

$$P(T_n > c_n | H_{(n)}) \to 1.$$

Note that (3.11) remains true if $t_o = 1 - n^{-\gamma}$ for any γ, $0 < \gamma < 1$.

Remark 3.1. The simulation results of Sen and Srivastava (1975) tend to support the above conclusions even for moderate sample sizes. They show that for $20 \leq n \leq 100$, $\{T_n\}$ is less powerful than $\{U_n\}$ when t_o is close to $\frac{1}{2}$ and more powerful when t_o is close to 0 or 1. When n is moderately large, the factor $\psi^2(t_o, \alpha, \beta)t_o(1-t_o)$ in the above efficiency calculation may dominate the $2\, \ell n_2\, n$ piece. Based on Sen and Srivastava's simulation results, it seems likely that for fixed α and β, $\psi^2(t_o, \alpha, \beta)t_o(1-t_o)$ increases as $|t_o - \frac{1}{2}| \to \frac{1}{2}$. Actually when $t_o \stackrel{\sim}{\sim} 1$, then $h(t) = 0(1 - t_o)$ and $\psi(t_o, \alpha, \beta)$ is of the order $(1 - t_o)^{-1}$ so that the factor $\psi(t_o, \alpha, \beta)t_o(1 - t_o)$ is of the order $(1 - t_o)^{-1}$. This argument gives a heuristic explanation as to why the sequence of tests $\{T_n\}$ is more powerful than $\{U_n\}$ when $(1 - t_o)$ (or t_o) converges to zero sufficiently fast.

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